Topic - Homogeneous Differential Equation of First order and First Degree.

Class - B.Sc II (Math-Hons) (Group B)

Date -

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Study Material - Homogeneous Equations -

A differential Equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where $f(x,y)$ and $g(x,y)$ are homogeneous function of x and y of the same degree is called Homogeneous Differential Equation

Working Rule

STEP I - put $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

STEP II - Diff. with respect to x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

STEP III - put $\frac{dy}{dx} = v + x\frac{dv}{dx}$

and $y = vx$ in the Given Equation

On doing so, the given differential Equation can be reduced to the form in which the variables are separable.

Problem :→ 1

Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

Solution :→ The Given Equation is

$x^2y\,dx - (x^3+y^3)\,dy = 0$

$\Rightarrow (x^3+y^3)\,dy = x^2y\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$ ——— (1)

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^3}$

$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v$

$x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^4}{1+v^3}$

$\Rightarrow \frac{dx}{x} = -\frac{1+v^3}{v^4}\,dv$

$\int\frac{dx}{x} = \int\left[-v^{-4} - \frac{1}{v}\right]dv$

$\log x = -\frac{v^{-3}}{} - \log v = $

$$\Rightarrow \log x = \frac{1}{3v^3} - \log v + c$$

$$\log x = \frac{x^3}{3y^3} - \log \frac{y}{x} + c$$

$$\log x = \frac{x^3}{3y^3} - \log y + \log x + c$$

$$\Rightarrow \log y = \frac{x^3}{3y^3} + c$$

Which is the required Solution

Ex:→ Solve.

$$x\,dy - y\,dx - \sqrt{x^2+y^2}\,dx = 0$$

Solution →

⇒ The Given differential Equation is

$$x\,dy - y\,dx - \sqrt{x^2+y^2}\,dx = 0$$

$$x\,dy = \left[y + \sqrt{x^2+y^2}\right]dx$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} \quad (1)$$

Put $y/x = v \Rightarrow y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting these values in (1) we have

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$$

On integrating

$$\int \frac{dx}{x} = \int \frac{dv}{\sqrt{1+v^2}}$$

$$\Rightarrow \log x = \log(v + \sqrt{1+v^2}) + \log c$$

$$\Rightarrow \log x = \log c(v + \sqrt{1+v^2})$$

$$\therefore x = c(v + \sqrt{1+v^2})$$

putting $v = y/x$

$$= c\left(\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}\right)$$

$$= c\left(\frac{y + \sqrt{x^2+y^2}}{x}\right)$$

$$\Rightarrow x^2 = c(y + \sqrt{x^2+y^2})$$

which is the required solution

Problem Solve

$$(x^2-y^2)\frac{dy}{dx} = 2xy$$

Solution – The given Equation is

$$(x^2-y^2)\frac{dy}{dx} = 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2} \quad \text{——— (1)}$$

Put $y = vx$ in (1)

and.

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

We have

$$v + x\frac{dv}{dx} = \frac{2x \cdot vx}{x^2 - v^2x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2v}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - v + v^3}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v + v^3}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(1+v^2)}{1-v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{1-v^2}{v(1+v^2)}\,dv$$

On integrating

$$\frac{dx}{x} = \int \frac{1-v^2}{v(1+v^2)}\,dv$$

$\therefore \int \frac{dx}{x} = \int \left(\frac{1}{v} - \frac{2v}{1+v^2} \right) dv$

$\Rightarrow \log x = \log v - \log(1+v^2)$

$\Rightarrow \log x = \log \frac{y}{x} - \log\left(1+\frac{y^2}{x^2}\right)$ put $v = y/x$

$\Rightarrow \log x = \log y - \log x - \log(x^2+y^2) + 2\log x + k$

$\Rightarrow \log x = \log y - \log(x^2+y^2) + \log x + k$

$\Rightarrow \log y = \log(x^2+y^2) + \log c$ put $k = \log c$

$\log y = \log c(x^2+y^2)$

$\therefore y = c(x^2+y^2)$

which is the required solution.